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On the Period Spectrum of a Symplectic Mapping

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Let M be a compact Riemannian manifold. It has been known for a long time that the singularities of the wave trace, trace$(\cos\sqrt{\Delta}t)$, are located at the periods of the closed geodesics. Do these singularities also contain information about the geometry of M in the neighborhood of a closed geodesic? We prove that the Birkhoff canonical form of the Poincaré map can be determined from the singularities of the wave trace.

1. INTRODUCTION

Let M be a manifold of dimension $2n+1$, α a contact form on M, and Ξ the corresponding contact vector field. (We recall that Ξ is defined by the identity, $\iota(\Xi)\,\alpha = 1$ and $\iota(\Xi)\,d\alpha = 0$.) Let

$$\exp t\Xi, \qquad -\infty < t < \infty, \tag{1.1}$$

be the one-parameter group of contact transformations generated by Ξ. Given a periodic trajectory, γ, of (1.1) we will denote by T_γ its period and by P_γ the linearized Poincaré map about γ. Let Π be the set of all periodic trajectories of the system (1.1) and let T and P be the maps

$$T\colon \Pi \to \mathbb{R} \tag{1.2}$$

and

$$P\colon \Pi \to \text{conjugacy classes of } Sp(n, \mathbb{R}) \tag{1.3}$$

defined by $\gamma \to T_\gamma$ and $\gamma \to P_\gamma$. This paper is concerned with the question: What sort of information about the system (1.1) can be extracted from the data (1.2)–(1.3)? For instance do such data enable one to decide whether two systems of type (1.1) are conjugate or not?

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Before we take up these questions we should point out that for (1.2) and (1.3) to be regarded as dynamical invariants of the system (1.1) one needs an *intrinsic* description of Π not depending on Ξ. Sometimes, this is possible. For instance, if X is a negatively curved Riemannian manifold and M is its unit cosphere bundle, there is a unique closed geodesic in each free homotopy class of loops on X, and hence T and P can be thought of as functions on the set of free homotopy classes of loops. In general, however, only the list of numbers

$$\{T_\gamma, \gamma \in \Pi\} \tag{1.4}$$

and the list of matrices

$$\{P_\gamma, \gamma \in \Pi\} \tag{1.5}$$

are, strictly speaking, dynamical invariants of (1.1). (In other words, T_γ and P_γ are invariants of (1.1) but the affixing of the label, γ, to them is not.) The fact that (1.4) and (1.5) are, as a computer scientist would say, "unformatted data" makes these invariants much less useful than (1.2)–(1.3). (For a discussion of this formatting problem see [KB].) However, in deformation problems, which is what we are mainly concerned with below, one can get around the labelling issue as follows: Assume, henceforth, that none of the matrices, P_γ, in the collection (1.5) have roots of unity in their spectrum. (In other words assume that all the periodic trajectories, $\gamma \in \Pi$, are non-degenerate.) Let α_s, $-\varepsilon < s < \varepsilon$, be a family of contact forms on M deforming α but *not* deforming the data (1.4) and (1.5). We claim that if M is compact there is a canonical bijective map

$$\Pi \cong \Pi_s$$

for all $s \in (-\varepsilon, \varepsilon)$: Let γ be in Π. Since the spectrum of P_γ does not contain roots of unity the inverse function theorem guarantees that for some subinterval, $(-\delta, \delta)$ of $(-\varepsilon, \varepsilon)$, there exists a unique family, $\gamma_s \in \Pi_s$, $s \in (-\delta, \delta)$, which depends smoothly on s, with $\gamma_0 = \gamma$. Moreover, since the deformation leaves the data (1.4) and (1.5) fixed,

$$T_\gamma = T_{\gamma_s} \qquad \text{and} \qquad P_\gamma = P_{\gamma_s}.$$

Thus, for all points $s \in (-\delta, \delta)$, γ_s is non-degenerate and has the same period, T_γ, as γ. Let $p_s \in \gamma_s$. By passing to a subsequence if necessary, we can assume that, as $s \to \delta$, p_s converges to a limit point, p. This limit point is clearly contained in a periodic trajectory, $\gamma_\delta \in \Pi_\delta$, of period T_γ. Moreover this periodic trajectory is non-degenerate since none of the matrices in the collection (1.5) have roots of unity as eigenvalues. By applying the inverse function theorem to γ_δ we see that the maximal interval for which γ_s is defined is both open and closed in $(-\varepsilon, \varepsilon)$ and hence has to be *all* of $(-\varepsilon, \varepsilon)$. Thus for all $s \in (-\varepsilon, \varepsilon)$ there are maps

$$\Pi \to \Pi_s$$

mapping γ to γ_s. An argument similar to that sketched above shows that these maps are bijective and hence provide a de facto labelling of Π_s by Π.

We now state a conjecture concerning (1.4) and (1.5) (to which we unfortunately know the answer in only a few isolated cases).

Conjecture. Every deformation of the system (1.1) which leaves the data (1.4)–(1.5) fixed is trivial: that is, there exist, for $s \in (-\varepsilon, \varepsilon)$, diffeomorphisms, $\Psi_s: M \to M$ depending smoothly on s such that Ψ_0 is the identity and $\Psi_s^* \alpha_s = \alpha$.

This conjecture is true if M is compact and the system (1.1) is Anosov (see [LMM]). However, for systems which are not Anosov, in particular, for systems of KAM type, there is not much evidence for this conjecture pro or con. The main result of this paper is a very weak version of this conjecture for KAM systems: Let $\gamma \in \Pi$ be an *elliptic* periodic trajectory of the system (1.1), i.e., suppose the eigenvalues of P_γ are all of modulus one. To be explicit suppose these eigenvalues are

$$\{e^{\pm 2\pi i w_k},\ k = 1, ..., n\}.$$

Suppose in addition that γ is non-degenerate in the sense that the numbers, $\{1, w_1, ..., w_n\}$, are linearly independent over the rationals. Finally, suppose that γ satisfies an appropriate Birkhoff twist condition (of which, more below). In Section 7 we prove the following:

THEOREM. *Given the hypotheses above, there exists a neighborhood, U, of γ and diffeomorphisms, $\Psi_s: (U, \gamma) \to (M, \gamma_s)$ the Ψ_s's depending smoothly on s (with Ψ_0 the identity) such that $\Psi_s^* \alpha_s$ is equal to α plus a term which vanishes to infinite order on γ.*

We give a brief sketch of how this theorem is proved: Suppose the flow (1.1) admits a global cross section, X (which will be true in our case with M replaced by a sufficiently small tubular neighborhood, U, of γ). Then X will be a $2n$-dimensional symplectic manifold, and the Poincaré map, $f: X \to X$, associated with (1.1) will be a symplectic mapping of X into itself. Consider now, in place of the system (1.1), the discrete dynamical system

$$\{f^n: X \to X,\ -\infty < n < \infty\} \tag{1.6}$$

We show in Section 2 that for a discrete dynamical system of type (1.6) one can, modulo some assumptions on X, define a period spectrum similar to

(1.4). This is the key ingredient in our proof and will play a key role in all that follows. In Section 3 we show, by means of a mapping cylinder construction, that there is a one–one correspondence between contact systems for which the one-parameter group, (1.1), admits a global cross section and discrete dynamical systems of type (1.6). This reduces the conjugacy problem for (1.1) to the (simpler) conjugacy problem for (1.6). What is even more important: it gives us two equivalent definitions of "period spectrum" for a discrete-time dynamical system; for we show that, under the correspondence that we have just described, the definition in Section 2 becomes the usual definition, (1.4). (A parenthetical remark: The mapping cylinder construction in Section 3 has been used in many contexts before, but our version of it for *contact* systems appears to be new.)

We mentioned above that in order to define the period spectrum of a discrete dynamical system we had to make some assumptions about X. These assumptions are that X be simply connected and that the symplectic form on X be exact. For the applications we have in mind these assumptions are too restrictive; so we show in Section 4 how one can, to a certain extent, get rid of them.

In Sections 2 and 3 we give two equivalent, but very different, definitions of period spectrum for a discrete dynamical system. If the system (1.6) is integrable there turns out to be a third equivalent definition which, in the two-dimensional case, is due to Colin de Verdière [C]. We describe this in Section 5. The main result of the paper is contained in Sections 6 and 7. It says, roughly speaking, that if the system (1.6) is "nearly integrable" the recipe for period spectrum described in Section 5 is "nearly correct." This result has two consequences. One is an asymptotic formula for the period spectrum of a "nearly integrable" system and the other is a procedure for reconstructing the Birkhoff canonical form of the system from its period spectrum. (We explain what we mean by "Birkhoff canonical form" in Section 6; but, for the moment the following provisional definition will do: For a system which is "nearly integrable" it is the integrable system to which we are comparing it.) Incidentally the material in Section 7 is, like the material in Section 5, largely motivated by the results obstrained by Colin de Verdière for two-dimensional systems in [C]. (We are also indebted to him for suggestions about how to extend his results to higher dimensions.)

We conclude with a few words about the material in the Appendix. This paper is partly based on lectures given by one of the authors to an audience of non-experts at the annual meeting of the AMS three years ago. In the spirit of those lectures we have tried to make this paper as non-technical as possible. However, we were required to make some rather delicate estimates on the iterates of the mapping (1.6) in order to show that, even after many iterations, its period spectrum and the period spectrum of the corresponding integrable system are not too far apart. The underlying ideas involved in these estimates are quite easy to comprehend but the technical details are messy. In order not to mar the readability of the paper we settled upon the (perhaps not ideal) expedient of relegating these estimates to the Appendix. As a consequence some of the details of the proof of Theorem 7.4 are duplicated in a slightly more technical form in the Appendix.

Acknowledgments. We have already mentioned our debt to Yves Colin de Verdière, both for the inspiration of his paper [C], from which we have taken several of the main ideas in Sections 5–7, and for his suggestions about how to extend these ideas to higher dimensions. Another person to whom we owe a debt of gratitude is Alan Weinstein. Several years ago he posed to us a facinating question about the spectral properties of Riemannian manifolds which, after many devious twists and turns, becomes the formal conjugacy question that we discussed above: Let X be a compact n-dimensional Riemannian manifold, let M be the unit cosphere bundle of X, let ι be the canonical imbedding of M into T^*X, and let $\alpha = \iota^*(\sum \xi_i \, dx_i)$. Weinstein's question has to do with the relationship between the one-parameter group of contact transformations, (1.1), associated with this contact structure and the corresponding "quantum object," the one-parameter group of unitary operators

$$\exp\sqrt{-1}t\sqrt{\Delta} \tag{1.7}$$

on $L^2(M)$. Let S be the set of periodic trajectories of the system (1.1) and for each $\gamma \in S$ let T_γ be the period of γ and P_γ the linear Poincaré map associated with γ, as in (1.2) and (1.3). The generalized Selberg trace formula ([DG, Sect. 4]; see also [C_1], [Ch]) says that

$$\operatorname{Re}(\operatorname{trace}(\exp\sqrt{-1}t\sqrt{\Delta})) = \sum e_\gamma(t), \qquad \gamma \in S, \tag{1.8}$$

where $e_\gamma(t)$ is a generalized function of t which is supported in a small neighborhood about T_γ and is smooth except at $t = T_\gamma$. Moreover, if $I - P_\gamma$ is invertible, the leading singularity of $e_\gamma(t)$ at $t = T_\gamma$ is an (uninteresting) constant times

$$|\det(I - P_\gamma)|^{-1/2} \, \delta(t - T_\gamma). \tag{1.9}$$

Thus T_γ and the absolute value of $\det(I - P_\gamma)$ are *spectral* invariants of X.

If none of the eigenvalues of P_γ are roots of unity the quantities

$$|\det(I - P_\gamma^N)|, \qquad N = \pm 1, \pm 2, ..., \tag{1.10}$$

are also spectral invariants of X (since the same reasoning applies to the iterates of γ); and, by a theorem of Stark, this sequence of numbers determines P_γ up to symplectic conjugacy (see [DG, Appendix]). Weinstein's question is: Is there an analogous result for the Poincaré map itself? It is known that if one subtracts (1.5) from $e_\gamma(t)$, the dominant term in the remainder is a multiple of Log $|t-T_\gamma|$; and if one subtracts *this* term, the dominant term is a constant multiple of $|t-T_\gamma|$. In general after stripping away k terms from $e_\gamma(t)$, one obtains a distribution whose dominant term is a constant, c_k times $|t-T_\gamma|^k$. Weinstein's conjecture is that these c_k's (and the c_k's associated with the iterates of γ) have encoded into them the Birkhoff canonical form of the Poincaré map associated with γ.

Unfortunately it has turned out to be very hard to compute even a few of the c_k's except in some special cases (see [ERT], [D]). However, it may be possible to get at the Birkhoff canonical form by a more circuitous route (if γ is elliptic): It is suspected that isospectral sets of metrics are not too pathological: that they are stratified sets with locally arc-wise connected components. Were this the case, the theorem above would say that the Birkhoff canonical forms associated with elliptic geodesics are *constant* along connected components of strata; and, therefore, the compactness results of [OPS] would imply that for any $\gamma \in \Pi$, the list of possibilities for the Birkhoff canonical form is *finite* in number.

2. Period Spectrum

Let W be a $2n$-dimensional symplectic manifold with symplectic form, ω, and let $f\colon W \to W$ be a symplectomorphism. Then f and its iterates

$$f^N, \qquad N = \pm 1, \pm 2, ..., \tag{2.1}$$

constitute a *discrete-time dynamical system*. In this section we describe some rather curious-looking numerical invariants associated with the periodic trajectories of this system. These invariants were defined in [GM], and, for billiard systems, studied in considerable detail in [MM] and [C_2]. Most of the material in this section is taken from [GM].

For the moment we will assume that ω is exact and W simply connected. (Both these assumptions can be considerably weakened, and we say a few words about how to do so in Section 4.)

Since ω is exact, there exists a one-form, α, such that $d\alpha = \omega$. Moreover, since $f^*\omega = \omega$, $f^*\alpha - \alpha$ is closed and hence, because W is simply connected, is exact: i.e., there exists a function, ϕ_α, such that

$$f^*\alpha - \alpha = d\phi_\alpha. \tag{2.2}$$

This function is, unfortunately, determined only up to an additive constant, but there are often natural ways of fixing this constant. For instance, if p_0 is a pre-assigned "origin" in W, we can fix this constant by requiring that $\phi_\alpha(p_0)=0$; or if W is of finite volume, another way to fix this constant is by requiring that $\int \phi_\alpha \omega^n = 0$. Let us see to what extent ϕ_α depends on α. If β is another one-form with the property that $d\beta=\omega$, then $\beta-\alpha$ is closed; so again, because of the simple-connectivity of W, there exists a smooth function, $\phi_{\alpha,\beta}$, such that $\beta-\alpha=d\phi_{\alpha,\beta}$, and hence, by (2.2)

$$\phi_\beta - \phi_\alpha = f^*\phi_{\alpha,\beta} - \phi_{\alpha,\beta}. \tag{2.3}$$

Now let

$$p_0 = p, \ldots, p_n = p = f(p_{n-1}) \tag{2.4}$$

be a periodic trajectory of f. By (2.3)

$$\sum \phi_\alpha(p_i) = \sum \phi_\beta(p_i); \tag{2.5}$$

i.e., the sum, (2.5), depends only on the trajectory, (2.4), not on the choice of α.

Definition. We will call (2.5) the "period" of the periodic trajectory, (2.4).[1] The map which assigns to every periodic trajectory its "period" we will call the *labelled period spectrum* of (2.1) and the image of this map the (unlabelled) *period spectrum.*

Now as in Section 1 let M be a $(2n+1)$-dimensional contact manifold, with contact form, α, and contact vector field, Ξ; and let γ be a periodic trajectory of the dynamical system (1.1). Let X be a $2n$-dimensional submanifold of M which intersects γ transversally at p_0 and let W be a small neighborhood of p_0 in X. If W is small enough, for every $p \in W$ the trajectory of (1.1) with initial point, p, will intersect X again approximately at time T_γ. Let us denote by $\phi(p)$ the *exact* time that this trajectory intersects X and by $f(p)$ the point of intersection.

Definition. The map

$$f\colon (W, p_0) \to (X, p_0) \tag{2.6}$$

which sends p to $f(p)$ is the *Poincaré map* associated with γ and the function

$$\phi\colon W \to \mathbb{R}, \qquad p \to \phi(p) \tag{2.7}$$

is the *return time* function.

[1] We use the notation "period" for (2.4) (cumbersome as this notation is) to distinguish (2.5) from what is usually called the period of the trajectory (2.4), viz., the integer n.

Let $\iota: X \to M$ be the inclusion mapping and let $\alpha_1 = \iota^*\alpha$. We will prove below the "Poincaré–Cartan" identity

$$f^*\alpha_1 = \alpha_1 + d\phi. \tag{2.8}$$

This identity has a number of implications: To begin with, since $d\alpha_1$ is symplectic in a small neighborhood of p_0 it says that the mapping (2.6) is symplectic. Moreover, if the point p and its iterates

$$p_1 = p, p_2 = f(p_1), ..., p_{N+1} = f(p_N) \tag{2.9}$$

are all in W and $p_{N+1} = p$, then the trajectory of (1.1) through p is periodic, and its "period" is equal to

$$\sum \phi(p_i) \tag{2.10}$$

since $\phi(p_i)$ is the interval of time between the ith and $(i+1)$st encounters of this trajectory with W. On the other hand if we set $\phi_\alpha = \phi$ in (2.2) we see that (2.10) is also the period of the periodic trajectory (2.9) of the system (2.1). Therefore, to summarize we have proved:

PROPOSITION 2.1. *Let f be the Poincaré map* (2.6) *and let* (2.1) *be the discrete-time dynamical system generated by f. Then the period spectrum of* (2.1) *is contained in the period spectrum of* (1.1).

Proof of the Identity (2.8). Let f_t: $W \to M$ be the map

$$f_t(p) = (\exp(t\phi(p))\, \Xi)(p), \tag{2.11}$$

and let μ be a k-form on M. Then

$$f_1^*\mu - f_0^*\mu = d\mathcal{Q}\mu + \mathcal{Q}d\mu, \tag{2.12}$$

where

$$Q\mu = \int_0^1 f_t^*(\iota(\dot{f}_t)\, \mu)\, dt. \tag{2.13}$$

(See, for example, [DeR].) However, by (2.11)

$$\dot{f}_t(p) = \phi(p)\, \Xi(f_t(p));$$

so, in particular, setting $\mu = \alpha$ and making use of the identities

$$\iota[\Xi]\, \alpha = 1$$

and

$$\iota(\Xi)\, d\alpha = 0,$$

we obtain from (2.12) and (2.13) the identity

$$f_1^* \alpha - f_0^* \alpha = d\phi,$$

which, in view of the fact that $f_0 = \iota$ and $f_1 = \iota \circ f$, is just the identity (2.8).

Q.E.D.

3. The Mapping Cylinder of a Symplectic Mapping

In this section we give, as an application of Proposition 2.1, an alternative definition of the period spectrum of a discrete-time dynamical system. First of all, however, to motivate our definition we will go over a standard construction in the theory of differentiable dynamical systems: the mapping-cylinder construction. For the moment we will forget about the fact that the mappings, (2.1), are symplectomorphisms; and consider them to be simply diffeomorphisms. Let

$$f^\natural \colon W \times \mathbb{R} \to W \times \mathbb{R}$$

be the diffeomorphism,

$$f^\natural(p, a) = (f(p), a + 1),$$

and let Γ be the group of diffeomorphisms of $W \times \mathbb{R}$ generated by $f^\natural$. Γ acts in a properly discontinuous fashion on $W \times \mathbb{R}$, so the quotient space

$$W_f \overset{\text{def}}{=} (W \times \mathbb{R})/\Gamma \tag{3.1}$$

is a differentiable manifold, and the point–coset correspondence

$$\pi \colon W \times \mathbb{R} \to W_f \tag{3.2}$$

is a smooth covering map.

Definition. We will call W_f the *mapping cylinder* of the diffeomorphism, f.

The one-parameter group of diffeomorphisms

$$\psi_t \colon W \times \mathbb{R} \to W \times \mathbb{R}, \qquad (w, a) \to (w, a + t), \tag{3.3}$$

commutes with $f^\natural$; so there exists a one-parameter group of diffeomorphisms

$$\phi_t \colon W_f \to W_f, \qquad -\infty < t < \infty \tag{3.4}$$

such that

$$\pi \circ \psi_t = \phi_t \circ \pi. \tag{3.5}$$

In addition the mapping

$$\iota \colon W \to W_f, \qquad w \to \pi(w, 0) \tag{3.6}$$

imbeds W into W_f as a codimension one submanifold of W_f; and for every point, $p \in W$, the trajectory of (3.4) through p hits W again at time $t = 1$ in the point, $f(p)$. Thus, the discrete-time dynamical system (2.1) can be thought of as a series of "cross-sectional slices" of the dynamical system, (3.4), at time intervals $t = 1, 2, \ldots$. (For more on this mapping cylinder construction, see [Sm].)

Suppose now that the system (2.1) has the property that it preserves a symplectic structure. We would like to be able to conclude that the system (3.4) has a similar kind of property. We will show that this is, in fact, the case provided that we make some trivial modifications in our definition of W_f. Here are the details:

As in Section 2 we will assume that W is simply connected and is equipped with a symplectic form, ω, which is exact. Let α be a one-form on W whose exterior derivative is ω and let ϕ_α be the function, (2.2). To simplify matters we will assume that ϕ_α is bounded and that the symplectic volume of W is finite, in which case we can normalize ϕ_α by requiring that

$$\int \phi_\alpha \omega^n$$

be equal to a prescribed constant. By choosing this constant large enough we can insure that

$$C^{-1} \leqslant \phi_\alpha \leqslant C \tag{3.7}$$

for some constant $C > 1$.

Now let $\tilde{f}_\alpha \colon W \times \mathbb{R} \to W \times \mathbb{R}$ be the diffeomorphism

$$\tilde{f}_\alpha(p, a) = (f(p), a + \phi_\alpha(p)) \tag{3.8}$$

and let Γ_α be the group of diffeomorphisms of $W \times \mathbb{R}$ generated by $\tilde{f}_\alpha$. Because of the property (3.7) this group acts in a properly discontinuous fashion on $W \times \mathbb{R}$, so the orbit space

$$W_{f,\alpha} = W \times \mathbb{R}/\Gamma_\alpha$$

is, as before, a differentiable manifold and the point–orbit mapping

$$\pi \colon W \times \mathbb{R} \to W_{f,\alpha} \tag{3.9}$$

is a C^∞ covering map. The one-parameter group of diffeomorphisms, (3.3), still commutes with $\tilde{f}_\alpha$; so there exists as before a one-parameter group of diffeomorphisms

$$(\phi_\alpha)_t\colon W_{f,\alpha} \to W_{f,\alpha} \tag{3.10}$$

such that $\pi \circ \psi_t = (\phi_\alpha)_t \circ \pi$. We leave (as an easy exercise) the following:

PROPOSITION 3.1. *There is a diffeomorphism of $W_{f,\alpha}$ onto W_f which carries orbits of* (3.10) *onto orbits of* (3.4).

So far we have not exploited the fact that f is a symplectomorphism. This fact, however, will now be used to prove the following result:

PROPOSITION 3.2. *$W_{f,\alpha}$ possesses a canonical contact form (which we will denote by $\alpha^{\#}$), and the contact vector field, Ξ_α, associated with $\alpha^{\#}$ is the infinitesimal generator of the one-parameter group* (3.10).

Proof. Consider the contact form

$$-(pr_1)^*\,\alpha + (pr_2)^*\,dt \tag{3.11}$$

on $W \times \mathbb{R}$. The mapping (3.8) and its iterates preserve this contact form; so there exists a contact form, $\alpha^{\#}$, on $W_{f,\alpha}$ such that $\pi^*\alpha^{\#}$ is the form (3.11).

Let us denote by $\partial/\partial t$ the infinitesimal generator of the one-parameter group (3.3). The interior product of $\partial/\partial t$ with the form (3.11) is one, and its interior product with the exterior derivative of the form (3.11) is zero; so $\partial/\partial t$ is (by definition) the contact vector field associated with the contact form (3.11); and the group (3.3) is the corresponding one-parameter group of contact transformations. Since $\pi \circ \psi_t = (\phi_\alpha)_t \circ \pi$, $(\phi_\alpha)_t$ is the one-parameter group of contact transformations associated with the corresponding contact structure on $W_{f,\alpha}$. Q.E.D.

We will next show that our construction does not really depend on the choice of α. Let β be another one-form on W such that $d\beta = \omega$. We will prove:

PROPOSITION 3.3. *There exists a canonical isomorphism of contact manifolds*

$$\gamma_{\alpha,\beta}\colon W_{f,\alpha} \to W_{f,\beta}. \tag{3.12}$$

Proof. Let $\phi_{\alpha,\beta}$ be as in (2.3) and let $g_{\alpha,\beta}$ be the mapping of $W \times \mathbb{R}$ onto $W \times \mathbb{R}$ sending (p, a) onto $(p, a + \phi_{\alpha,\beta}(p))$. Then

$$\begin{aligned} g^*_{\alpha,\beta}(-pr_1^*\beta + (pr_2)^*\,dt) &= (pr_1)^*\,(-\beta + d\phi_{\alpha,\beta}) + (pr_2)^*\,dt \\ &= -(pr_1)^*\,\alpha + (pr_2)^*\,dt. \end{aligned}$$

Moreover,

$$\tilde{f}_\beta \circ g_{\alpha,\beta}(p, t) = (f(p), t + \phi_\beta(p) + \phi_{\alpha,\beta}(p))$$

and

$$g_{\alpha,\beta} \circ \tilde{f}_\alpha(p, t) = (f(p), t + \phi_{\alpha,\beta}(f(p)) + \phi_\alpha(p)).$$

However, by (2.3)

$$f^*\phi_{\alpha,\beta} - \phi_{\alpha,\beta} = \phi_\beta - \phi_\alpha,$$

so

$$\tilde{f}_\beta \circ g_{\alpha,\beta} = g_{\alpha,\beta} \circ \tilde{f}_\alpha.$$

Thus $g_{\alpha,\beta}$ induces a mapping of $W_{f,\alpha}$ onto $W_{f,\beta}$ which carries the contact form on $W_{f,\alpha}$ onto the contact form on $W_{f,\beta}$. Q.E.D.

This result justifies the following:

DEFINITION–THEOREM 3.4. *There exists a unique universal object, W_f, called the* mapping cylinder of f, *with the following properties*:

(a) *W_f is a $(2n+1)$-dimensional contact manifold.*

(b) *For every choice of a one-form, α, on W with the property $d\alpha = \omega$ there exists an isomorphism of contact manifolds*

$$\gamma_\alpha \colon W_f \xrightarrow{\cong} W_{f,\alpha}. \tag{3.13}$$

(c) *If β is another such one-form, the diagram*

$$\begin{tikzcd} & W_f \arrow[dl, "\gamma_\alpha"'] \arrow[dr, "\gamma_\beta"] & \\ W_{f,\alpha} \arrow[rr, "\gamma_{\alpha,\beta}"] & & W_{f,\beta} \end{tikzcd}$$

commutes.

In the classical construction of the mapping cylinder which we outlined at the beginning of this section, the discrete-time dynamical system (2.1) can be thought of as a series of "cross-sectional slices" of the dynamical system (3.4) at time intervals $t = 1, 2, \ldots$. Here the situation is a little bit more complicated:

Let Ξ be the contact vector field associated with the canonical contact form on W_f and let

$$\exp t\Xi, \qquad -\infty < t < \infty \tag{3.14}$$

be the one-parameter group of contact transformations that it generates.

For every choice of a one-form, α, whose exterior derivative is ω we obtain an imbedding

$$\iota\colon W \to W_f, \qquad \iota(p) = \gamma_\alpha^{-1}(\pi(p, o)); \tag{3.15}$$

however, it is no longer true that for all $p \in W$ the trajectory of (3.14) through p returns to W precisely at $t = 1$. What we can say, however, in view of (3.7) is that this trajectory *does* not return to W before time $t = C^{-1}$ and that it *does* return to W at some time between $t = C^{-1}$ and $t = C$. Furthermore, when it returns for the first time the point where it intersects W is $f(p)$. In other words

PROPOSITION 3.5. *The mapping $f\colon W \to W$ is the (global) Poincaré map associated with* (3.14).

From this result and Proposition 2.1 we deduce the main result of this section:

THEOREM 3.6. *The period spectrum of the system* (2.1) *is identical with the period spectrum of* (3.14).

This theorem can be taken as an alternative definition of the period spectrum of a discrete-time dynamical system.

We will conclude this section by proving a result which we will need in Section 7. Let M be a contact manifold with contact form, α^M, and contact vector field, Ξ^M, and let

$$\exp t\Xi^M, \qquad -\infty < \infty \tag{3.16}$$

be the one-parameter group of contact transformations generated by Ξ^M. Let W be a codimension one submanifold of M and C a constant greater than one such that the following are true:

(a) Every point in M lies on a trajectory of (3.16) whose initial point is on W.

(b) For every point, $p \in W$, the trajectory of (3.16) with initial point at p intersects W transversally at time $t = 0$. (3.17)

(c) The trajectory in (b) does not intersect W between times $t = 0$ and $t = C^{-1}$.

(d) The trajectory in (b) *does* intersect W at some time between $t = C^{-1}$ and $t = C$.

To say that a W with these properties exists is equivalent to saying that there is a global Poincaré map

$$f\colon W \to W \tag{3.18}$$

defined by the flow, (3.16). Suppose now that there exists a W with these properties.

THEOREM 3.7. *There is a natural diffeomorphism of M onto W_f which maps $-\alpha^M$ onto the canonical contact form on W_f.*

(In other words if there exists a submanifold of M for which a global Poincaré map exists, one can "reconstruct" M from this Poincaré map in a canonical way.)

Proof. For every point $p \in W$ let $\phi(p)$ be the time required for the trajectory of (3.16) through p to return to W, and let α be the restriction of α_M to W. Then by (2.7)

$$f^*\alpha = \alpha + d\phi \tag{3.19}$$

and, by (3.17), properties (c) and (d),

$$C^{-1} \leqslant \phi \leqslant C. \tag{3.20}$$

Now let

$$h\colon W \times \mathbb{R} \to M \tag{3.21}$$

be the mapping which maps (p, t) onto $\exp(-t\Xi^M)(p)$. This map maps the points (p, a) and $(f(p), \phi(p) + a)$ onto the same point of M, so it satisfies

$$h \circ f^{\natural} = h;$$

and hence gives rise to a well-defined map,

$$h_1\colon W_{f,\alpha} \to M. \tag{3.22}$$

It is easy to check that h_1 is bijective (in fact, is a *diffeomorphism* of $W_{f,\alpha}$ onto M).

Coming back to h: by construction, h intertwines the one-parameter group (3.2) and the one-parameter group $\exp(-t\Xi^M)$; therefore the infinitesimal generator of the one-parameter group (3.2), which we have been denoting $\partial/\partial t$, is mapped by h onto $-\Xi^M$. Thus, in particular,

$$\iota\left(\frac{\partial}{\partial t}\right) h^*\alpha_M = -1$$

and

$$\iota\left(\frac{\partial}{\partial t}\right) h^* d\alpha_M = 0.$$

Let

$$\alpha^\natural = -h^*\alpha_M - (pr_2)^*\, dt. \tag{3.23}$$

Then we can rewrite these equations in the form

$$\iota\left(\frac{\partial}{\partial t}\right)\alpha^\natural = \iota\left(\frac{\partial}{\partial t}\right) d\alpha^\natural = 0.$$

These equations say that the one-form, $\alpha^\natural$, is *basic* with respect to the fibration

$$pr_1\colon W \times \mathbb{R} \to W;$$

that is, there exists a one-form, α_1, on W such that

$$\alpha^\natural = (pr_1)^*\, \alpha_1.$$

However, denoting by ι_0 the inclusion of W into $W \times \mathbb{R}$ sending p to $(p, 0)$, we have, by definition,

$$\begin{aligned}\alpha_1 &= \iota_0^*(pr_1)^*\, \alpha_1 = \iota_0^*\alpha^\natural = -\iota_0^* h^*\alpha_M = -\iota^*\alpha M\\ &= -\alpha.\end{aligned}$$

Thus, by (3.23).

$$-h^*\alpha_M = -(pr_1)^*\, \alpha + (pr_2)^*\, dt.$$

The form on the right is just the pull-back to $W \times \mathbb{R}$ of the canonical contact form on $W_{f,\alpha}$; so the map (3.22) maps this canonical contact form onto $-\alpha_M$, as claimed.

4. Other Definitions of Period Spectrum

In our definition of the period spectrum of the discrete-time dynamical system (2.1) we made two assumptions about W: simple-connectivity and exactness of the symplectic form. Unfortunately, in many cases of interest (including the example that we discuss in the next section) the first assumption does not hold, and there are also cases of interest for which the second does not either. (For instance, if W is compact, ω cannot be exact since $[\omega^n](W)$ is the symplectic volume of W.) In this section we show how both these assumptions can be weakened if one is willing to make some additional assumptions about f.

For the moment we will continue to assume that the symplectic form is exact; however, instead of assuming that W is simply connected, we will assume that f is homotopic to the identity. If this is the case we can still find a function, ϕ_α, for which (2.2) holds, and hence, for every periodic trajectory

$$\gamma = \{p_1, ..., p_N\} \tag{4.1}$$

we can still define its "period" to be the sum

$$P(\gamma, \alpha) = \sum \phi_\alpha(p_i). \tag{4.2}$$

We will see shortly, however, that this expression is no longer independent of α. To see to what extent it depends on α, we will need some results about "winding numbers": Let c be an element of $H^1(W, \mathbb{R})$ and let μ be a closed one-form representing c. Since f is homotopic to the identity, there exists a smooth function, ϕ_μ, such that

$$f^*\mu - \mu = d\phi_\mu. \tag{4.3}$$

(This function is determined only up to an additive constant; however, we will assume, as in Section 2, that some scheme exists for specifying this constant.) Now consider the sum

$$\sum \phi_\mu(p_i) \tag{4.4}$$

over the points of the trajectory (4.1). We claim that (4.4) depends only on c, not on μ. In fact if one chooses another one-form, ν, in the same cohomology class, there exists a function, $\phi_{\mu,\nu}$, such that

$$\nu - \mu = d\phi_{\mu,\nu}$$

and hence

$$\phi_\nu - \phi_\mu = f^*\phi_{\mu,\nu} - \phi_{\mu,\nu};$$

so the two sums

$$\sum \phi_\mu(p_i) \qquad \text{and} \qquad \sum \phi_\nu(p_i)$$

are equal.

DEFINITION 4.1. Let γ be the periodic trajectory, (4.1), of the system (2.1). We will define the sum (4.4) to be the *winding number* of γ with respect to c and denote it by $W(\gamma, c)$.

Remark. One can also define $W(\gamma, c)$ in a more geometric way: Let

$$\pi\colon W_1 \to W$$

be the universal covering space of W. Since f is homotopic to the identity it lifts to a diffeomorphism, $f_1\colon W_1 \to W_1$. Moreover, since W_1 is simply connected, $\pi^*\mu$ is exact; so there exists a C^∞ function, ψ_μ, on W_1 such that $\pi^*\mu = d\psi_\mu$. Thus

$$\begin{aligned}\pi^* d\phi_\mu &= \pi^*(f^*\mu - \mu) = f_1^*\pi^*\mu - \pi^*\mu\\ &= d(f_1^*\psi_\mu - \psi_\mu);\end{aligned}$$

hence, for some constant, k,

$$\pi^*\phi_\mu = f_1^*\psi_\mu - \psi_\mu + k. \tag{4.5}$$

Now let q_1 be a point in W_1 above p_1 and let $q_2, q_3, \ldots,$ be its iterates with respect to f_1. Then

$$\gamma_1 = \{q_1, \ldots, q_N, \ldots,\} \tag{4.6}$$

is a trajectory of the dynamical system, $\{f_1^N, N = \pm 1, \ldots,\}$ and π maps it onto the periodic trajectory, (4.1); however, (4.6) is no longer necessarily periodic, i.e., its end-points, q_1 and q_N, are not necessarily equal. We claim:

$$W(\gamma, c) = \psi_\mu(q_N) - \psi_\mu(q_1) + (N+1)\,k. \tag{4.7}$$

Proof. Sum both sides of (4.5) over $q_1, \ldots, q_{N-1}$. The left-hand sum is, by definition, $W(\gamma, c)$, and, on the right, all terms cancel except the first and last terms, and one is left with (4.7). Q.E.D.

Now as in Section 2 let α and β be one-forms whose exterior derivatives are both equal to ω. Then $\beta - \alpha$ is closed, so it determines a cohomology class, c, in $H^1(W, \mathbb{R})$. We will prove that for every periodic trajectory, $\gamma = \{p_1, \ldots, p_N\}$, of (2.1):

$$P(\gamma, \beta) = P(\gamma, \alpha) + W(\gamma, c). \tag{4.8}$$

Proof. Let

$$f^*\beta - \beta = d\phi_\beta \qquad \text{and} \qquad f^*\alpha - \alpha = d\phi_\alpha \tag{4.9}$$

and let μ be a one-form representing the cohomology class, c. Then there exists a function, $\phi_{\alpha,\beta}$, such that

$$\beta - \alpha = \mu + d\phi_{\alpha,\beta}. \tag{4.10}$$

Let ϕ_μ be a function satisfying (4.3). Then from (4.3), (4.9), and (4.10) one obtains

$$\phi_\beta = \phi_\alpha + \phi_\mu + f^*\phi_{\alpha,\beta} - \phi_{\alpha,\beta}, \tag{4.11}$$

and, by summing both sides of this identity over γ, one obtains (4.8).

Q.E.D.

EXAMPLE. Let W be an annulus in the plane and f an area preserving mapping of W onto itself which is homotopic to the identity. Then $H^1(W, \mathbb{R})$ is one-dimensional and is generated by $[d\theta]$. Thus, the coholomology class, c, of $\beta - \alpha$ has to be a multiple, λ, of $[d\theta]$, and since

$$f^*\, d\theta - d\theta = d(\arg f),$$

(4.8) reduces to

$$P(\gamma, \beta) = P(\gamma, \alpha) + \lambda W(\gamma),$$

where $W(\gamma)$ is the usual winding number of the trajectory, γ, around the origin.

We will next discuss a notion of period spectrum for symplectic manifolds for which the symplectic form is *not* exact. In this case there does not appear to be any natural way of defining the "period" of a periodic trajectory of the system (2.1) as a numerical invariant; however, it is sometimes possible to define it as an $\mathbb{R} \bmod \mathbb{Z}$ invariant.

Suppose, for instance, that ω is *integral*, i.e., suppose that the ReRham cohomology class, $[\omega]$, in $H^2(W, \mathbb{R})$ is the image of a cohomology class in $H^2(W, \mathbb{Z})$. Then there exists a Hermitian line bundle, L, and a connection, ∇, on L such that ω is the curvature form of ∇. If f is a symplectomorphism of W, then, modulo some topological assumptions which we will not go into here, f can be lifted to a connection-preserving automorphism of L. (For instance, if W is simply connected, (L, ∇) is determined uniquely, up to isomorphism, by ω, and a lifting of f exists and is unique up to multiplication by a constant of modulus one.) Let us denote such a lifting by $f^\#$. Then if $p = p_0, ..., p_n = p = f(p_{n-1})$, is a periodic trajectory of f, $(f^\#)^n_p = c(\mathrm{Id})_p$, on the fiber, L_p, of L at p, where c is a constant of modulus one. One can define the "period" of the trajectory above to be arg $c/2\pi$. If W is simply connected, this quantity is intrinsically defined for all periodic trajectories of period n, up to addition of a fixed constant $n\Theta_0$ (not depending on the trajectory) and is an $\mathbb{R} \bmod \mathbb{Z}$ invariant of the trajectory. (If W is not simply connected, one has to be careful about the choices of L and the lifting in order to make this quantity well-defined: however, this is very often possible.)

EXAMPLE. Let $W = \mathbb{R}^2/\mathbb{Z}^2$, with its standard symplectic form, and let f be the self-map of W associated with a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of determinant one with integer entries. The period spectrum of f turns out to be well-defined in the sense described above and, in fact, turns out to be computable as follows: Let p be a periodic point of order n, i.e., a point that returns to itself after n iterations. This means that if v is a vector representing p in $\mathbb{R}^2$, then

$$A^n v = v + z, \tag{4.12}$$

with $z \in \mathbb{Z}^2$. Let Ω be the standard symplectic form on $\mathbb{R}^2$.

CLAIM. *The "period,"* mod $\mathbb{Z}$, *of the periodic trajectory of period n starting at p is*

$$\Omega(v, A^n v). \tag{4.13}$$

We will not give the proof here; however, we will show that (4.13) is well-defined as a mod $\mathbb{Z}$ invariant. Suppose one replaces v by another vector, v_1, representing p, i.e., $v_1 = v + w$, where w is in $\mathbb{Z}^2$. Then

$$\begin{aligned} A^n v_1 &= A^n v + A^n w = v + z + A^n w \\ &= v_1 + z + A^n w - w. \end{aligned}$$

Hence, modulo $\mathbb{Z}$,

$$\begin{aligned} \Omega(v_1, A^n v_1) &= \Omega(v_{1,z} + A^n w - w) \\ &= \Omega(v, z) + \Omega(v, A^n w - w). \end{aligned}$$

The first term on the right is just $\Omega(v, A^n v)$, and the second term can be rewritten in the form $\Omega(A^{-n} v - v, w)$. By (4.12), $A^{-n} v - v$ is in $\mathbb{Z}^2$; so this term is in $\mathbb{Z}$. Hence, modulo $\mathbb{Z}$, the quantity, (4.13), is well-defined independently of v. (We will discuss this example (and examples like it) in more detail in a sequel to this article.)

5. THE INTEGRABLE CASE

The system, (2.1), is *integrable* if there exist n independent functions on W which are in involution and are integrals of motion of the system.

Formulated more geometrically, this means that there exist an n-dimensional manifold, B, and a C^∞ fibration

$$\pi\colon W \to B \tag{5.1}$$

such that

$$\pi \circ f = \pi, \tag{5.2}$$

and such that the fibers of (5.1) are Lagrangian submanifolds of W. We will assume, in addition, that the fibers of (5.1) are compact and connected, and that B is contractible. We will also continue to assume that ω is exact. These assumptions enable one to define *action–angle coordinates* on W. More explicitly, by a theorem of Arnol'd, the fibers of π have to be n-tori; so if F_b is the fiber above $b \in B$, there is an isomorphism,

$$H_1(F_b, \mathbb{Z}) \cong \mathbb{Z}^n. \tag{5.3}$$

The isomorphism, (5.3), is not canonical; however, if one specifies (5.3) at some point, b_0, then, because of the simple-connectivity of B, (5.3) is then specified at *all* points. This implies that there exist closed curves

$$\gamma_k(b), \qquad k = 1, ..., n, \tag{5.4}$$

on F_b which vary smoothly as b varies and which are generators of $H_1(F_b, \mathbb{Z})$. Now let α be a one-form whose exterior derivative is ω, and let $I_k(b)$ be the integral of α over $\gamma_k(b)$. The I_k's form a system of coordinates on B called *action coordinates*. If one replaces the one-form, α, by a one-form, β, whose exterior derivative is also equal to ω, then $\beta - \alpha$ is closed; so the integral of β over $\gamma_k(b)$ differs from the integral of α by a quantity which does not depend on b. Hence the effect of replacing β by α is to change I_k by an additive constant.

Another ambiguity in the definition of the I_k's is the choice of the map (5.3). As we pointed out above, however, this choice is ambiguous only in the sense that one has to specify (5.3) at a single point; so the only way to change (5.3) is by applying to $\mathbb{Z}^n$ a linear transform of the form

$$n'_k = \sum a_{k,l} n_l,$$

where $(a_{k,l}) \in GL(n, \mathbb{Z})$; and this has the effect of changing the I_k's by a transform of the form

$$I'_k = \sum a_{k,l} I_l.$$

Thus, to summarize, the I_k's are intrinsically defined up to the choice of a $c \in \mathbb{R}^n$ and an $n \times n$ matrix, $(a_{k,l})$, with integer entries.

Given the action coordinates, $I_1, ..., I_n$, one can find functions

$$\theta_i, \qquad i = 1, ..., n, \tag{5.5}$$

on W which are defined modulo $2\pi\mathbb{Z}$ and satisfy

$$\omega = \sum d\theta_i \wedge dI_i. \tag{5.6}$$

These functions are called angle coordinates; and, together with the action coordinates, they form a Darboux coordinate system on W. Unlike the action coordinates they are *not* intrinsically defined. In fact given a smooth function, $F = F(I_1, ..., I_n)$, the coordinates

$$\theta'_i = \theta_i - \frac{\partial F}{\partial I_i}, \qquad i = 1, ..., n, \tag{5.7}$$

also satisfy (5.6). However, one can show that if $\theta_1, ..., \theta_n$ and $\theta'_1, ..., \theta'_n$ are two sets of angle coordinates, they have to be related by a transformation of the form (5.7). (For more on action–angle coordinates see [Arn] or [Dui].)

Suppose now that f is a symplectomorphism of W satisfying (5.2). Let $I_1, ..., I_n, \theta_1, ..., \theta_n$, be a system of action–angle coordinates on W. Then

$$f^* I_k = I_k, \qquad k = 1, ..., n; \tag{5.8}$$

so, by (5.6), the $f^*\theta_k$'s are also a system of angle coordinates. Thus, by (5.7), there exists a smooth function, $F = F(I_1, ..., I_n)$, such that

$$f^*\theta_k = \theta_k + \frac{\partial F}{\partial I_k}, \qquad k = 1, ..., n. \tag{5.9}$$

Note, by the way, that since the action coordinates are intrinsically defined up to additive constants, so is the function, F. The mapping, (5.8)–(5.9), is called *non-degenerate* if, for all $b \in B$,

$$\det\left(\frac{\partial^2 F}{\partial I_i \, \partial I_j}\right) \neq 0 \tag{5.10}$$

The main result of this section is a theorem which says, roughly speaking, that the period spectrum of the system (2.1) is easy to compute when f is non-degenerate, i.e., when (5.10) holds. Before stating this theorem, however, we first recall a few elementary facts about Legendre transforms. Let

$$\mathscr{L}_F: B \to \mathbb{R}^n \tag{5.11}$$

be the Legendre transform associated with F. (By definition

$$\mathscr{L}_F(b) = \left(\frac{\partial F}{\partial I_1}(b), ..., \frac{\partial F}{\partial I_n}(b)\right).$$

(See, for instance, [Arn].)

Since (5.10) holds, the image of $\mathscr{L}_F$ is an open subset, U, of $\mathbb{R}^n$ and $\mathscr{L}_F$ is locally a diffeomorphism of B onto U. We will stengthen the assumption (5.10) by assuming that $\mathscr{L}_F$ is *globally* a diffeomorphism of B onto U.

PROPOSITION. *The inverse of $\mathscr{L}_F$ is also a Legendre transform. In fact*

$$\mathscr{L}_F^{-1} = \mathscr{L}_H, \tag{5.12}$$

where

$$H(q) = \left(I_1 \frac{\partial F}{\partial I_1} + \cdots + I_n \frac{\partial F}{\partial I_n} - F\right) \mathscr{L}_F^{-1}(q). \tag{5.13}$$

Proof. See the previous reference or [GS, p. 397].

Suppose now that q is a point in U with coordinates

$$q_i = \frac{2\pi r_i}{N}, \qquad i = 1, ..., n, \tag{5.14}$$

where $r_1, ..., r_n$ and N are integers. Without loss of generality we can assume that the r_i's and N have no common multiples. Let $b = \mathscr{L}_H(q)$, and consider the mapping, (5.8)–(5.9), on the fiber above b. By (5.9) this mapping is just translation by q; so if we iterate it N times we obtain the identity mapping. Thus, every point, p, on the fiber above b is periodic of period N. Moreover, as the system, (2.1), moves along the periodic trajectory through p, the angle variable, θ_i, winds r_i times around the unit circle; so, to summarize, if one denotes by c_i the DeRham cohomology class in $H^1(W\mathbb{R})$ defined by $d\theta_i$, *the trajectory*, γ, *through p is periodic of period N and has winding numbers*

$$w(\gamma, c_i) = r_i, \qquad i = 1, ..., n. \tag{5.15}$$

Note, by the way, that the converse of this statement is also true: if p is a periodic point of period N, and the trajectory, γ, through p has winding numbers, (5.15), then the coordinates of the point

$$q = (\mathscr{L}_F \circ \pi)(p)$$

are given by (5.14).

Let us now compute the period spectrum of the system (2.1). Let $\alpha = \sum I_k \, d\theta_k$. Thus, by (5.8) and (5.9)

$$f^*\alpha - \alpha = \sum I_k \, d\left(\frac{\partial F}{\partial I_k}\right) = d\left(\sum I_k \frac{\partial F}{\partial I_k} - F\right).$$

On the other hand, by (2.2), the right-hand side is $d\phi_\alpha$; so

$$\phi_\alpha = \sum I_k \frac{\partial F}{\partial I_k} - F.$$

Thus if $q = (\mathscr{L}_F \circ \pi)(p)$, $\phi_\alpha(p) = H(q)$. In particular, ϕ_α is constant along the trajectories of (2.1); so if the trajectory through p is periodic of period N, the sum (2.5) is just $NH(q)$. Thus, to summarize, we have proved:

THEOREM. *Let q be the point in U with coordinates* (5.14) *and let p be any point on the fiber above $\mathscr{L}_H(q)$, Then the trajectory, γ, through p is periodic of period N and its "period" is*

$$P(\gamma, \alpha) = NH(q). \tag{5.16}$$

The set of points, (5.14), *is dense in U; so, in view of* (5.8), (5.9) *this theorem has, as a corollary, the following "inverse spectral" result.*

THEOREM. *The labelled period spectrum of the system* (2.1) *determines* (2.1) *up to symplectomorphism.*

Remark. The *unlabelled* period spectrum, by contrast, tells one relatively little about the mapping, f.

6. The Non-integrable Case

If one perturbs an integrable system of the type described in Section 5, the tori, F_b, associated with the points, (5.14), break up into a finite number of periodic trajectories, each having the same period (i.e., N) and the same winding numbers (viz., $r_1, ..., r_n$) as the trajectories on F_b of the unperturbed system. In Section 7 we show that in certain cases there are asymptotic formulas for the "periods" of these trajectories as $N \to \infty$ and the r_i's stay bounded. The proof of this fact will require a close inspection of the classical proof (by Poincaré, Birkhoff *et al.*) of the existence of these trajectories. This we will do in the six subsections below:

1. Let M be a symplectic manifold and let $h: M \times I \to M$ be an isotopy with the property that, for every $t \in I$, $h_t: M \to M$ is a symplectomorphism. This isotopy is said to be *Hamiltonian* if, for every $t \in I$, the vector field

$$\Xi_t = f_t^{-1} \frac{df_t}{dt}$$

is globally Hamiltonian. Using this notion we will introduce an equivalence relation on the set of all Lagrangian submanifolds of M. Given two Lagrangian submanifolds, Λ and Λ_1, we will say (following Weinstein [W]) that they are *isodrastic* if there exists a Hamiltonian isotopy, h, such that h_0 is the identity, and h_1 maps Λ diffeomorphically onto Λ_1.

2. Let us see what it means for two nearby Lagrangian manifolds to be isodrastic. Let Λ be a *compact* Lagrangian submanifold of M. By the Weinstein tubular neighborhood theorem, there exist a neighborhood, U, of Λ in M, a neighborhood, U_0, of the zero section in $T^*\Lambda$, and a symplectomorphism, Φ: $U \to U_0$, which conjugates the inclusion mapping $\iota: \Lambda \to M$ and the standard inclusion mapping $\iota_0: \Lambda \to T^*\Lambda$, $p \to (p, 0)$. Now let Λ_1 be a Lagrangian submanifold of M which is "C^1-close" to Λ in the sense that the isotopy, h, in the paragraph above can be chosen C^1-close to the identity isotopy. Via the symplectomorphism, Φ, we can think of Λ as sitting inside of U_0 as the graph of a section

$$s: \Lambda \to T^*\Lambda. \tag{6.1}$$

Let μ be the one-form on Λ defined by s. Since Λ_1 is Lagrangian, μ has to be closed. We claim (see [W]).

THEOREM 6.1. *Λ and Λ_1 are isodrastic if and only if μ is exact.*

If Λ and Λ_1 are isodrastic, there exists a smooth function

$$\phi: \Lambda \to \mathbb{R}, \qquad \mu = d\phi$$

called *the defining function of Λ_1 relative to* (U, Φ). Note that, by (6.1), the critical points of ϕ are exactly the points where Λ and Λ_1 intersect. Since Λ is compact, ϕ must have at least two critical points, and hence Λ and Λ_1 must have at least two points of intersection.

3. For applications to the fixed point theorems we will need to "functorialize" the result above: Let M and N be symplectic manifolds and let Γ be a canonical relation between M and N (i.e., a Lagrangian submanifold of $M \times N^-$). Let Λ be a compact Lagrangian submanifold of M. For the following see [Hör].

THEOREM 6.2. *Suppose $pr_1 \colon \Gamma \to M$ is transversal to Λ. Then the set*

$$\Gamma \circ \Lambda = \{q \in N, \exists p \in \Lambda \text{ s.t. } (p, q) \in \Gamma\}$$

is a Lagrangian submanifold of N.

Now let Λ_1 be a Lagrangian submanifold of M such that Λ and Λ_1 are isodrastic. Let h be a Hamiltonian isotopy joining Λ to Λ_1 and, for all $t \in I$, let $\Lambda_t = h_t(\Lambda)$.

THEOREM 6.3. *Suppose that $pr_1 \colon \Gamma \to M$ is transversal to Λ_t for all t. Then all the Lagrangian manifolds, $\Gamma \circ \Lambda_t$, are isodrastic.*

(In particular, $\Gamma \circ \Lambda$ and $\Gamma \circ \Lambda_1$ are isodrastic.)

4. Let M be a symplectic manifold of dimension $2n$, let S be a manifold of dimension n, and let $\mathfrak{p} \colon M \to S$ be a fibration whose fibers are Lagrangian submanifolds of M. For every $s \in S$ let M_s be the fiber above s. Let $f \colon M \to M$ be a symplectic mapping with the property that, for every $s \in S$,

$$f \colon M_s \to M$$

is transversal to M_s. Then if p_s is a point of intersection of M_s and $f(M_s)$, it is isolated (there is a neighborhood of p_s containing no other points of intersection) and depends smoothly on s. In other words:

THEOREM 6.4. *The set*

$$\Lambda_f = \{p \in M, \mathfrak{p}(p) = \mathfrak{p}(f(p))\} \tag{6.2}$$

is an n-dimensional submanifold of M, and the map, $\mathfrak{p} \colon \Lambda_f \to S$ is locally a diffeomorphism of Λ_f into S.

We will now prove a slightly less trivial result:

THEOREM 6.5. *Λ_f is a Lagrangian submanifold of M.*

Proof. We apply Theorem 6.2 with M replaced by $M \times M^-$, with N replaced by M, with Γ replaced by the set,

$$\Gamma_1 = \{(p, q, p) \in (M \times M^-) \times M,\ \mathrm{p}(p) = \mathrm{p}(q)\},$$

and with Λ replaced by graph f. It is easy to check that

(a) Γ_1 is a canonical relation between $M \times M^-$ and M.

(b) The map, $pr_1 \colon \Gamma_1 \to M \times M^-$, is transversal to graph f if and only if, for all $s \in S$ the map, $f \colon M_s \to M$, is transversal to M_s.

(c) Λ_f is equal to Γ_o(graph f).

Thus we can apply Theorem 6.2 to Γ_1 and graph f; and, in view of c, this proves Λ_f is Lagrangian.

5. Let f_1 be a symplectic mapping of M onto itself, which is a "small" perturbation of f. We will say that f and f_1 are isodrastic if their graphs are isodrastic as Lagrangian submanifolds of $M \times M^-$. If f_1 is a "sufficiently" small perturbation of f, it also satisfies the hypotheses of Theorem 6.4, and hence Λ_{f_1} is also a Lagrangian submanifold of M. Therefore, if f and f_1 are isodrastic, Theorem 6.3 implies the following elementary (but important) result.

THEOREM 6.6. *The Lagrangian manifolds, Λ_f and Λ_{f_1}, are isodrastic.*

In particular suppose that Λ_f is mapped onto itself by f. Then

$$\Lambda_{f_1} \sim \Lambda_f = f(\Lambda_f) \sim f(\Lambda_{f_1}) \sim f_1(\Lambda_{f_1}).$$

The relation, "$\sim$," is "isodrasticity." (The last two equivalences are easily deduced from Theorem 6.3.) Hence (since isodrasticity is an equivalence relation)

$$\Lambda_{f_1} \sim f_1(\Lambda_{f_1}). \tag{6.3}$$

Suppose now that Λ_f is connected and compact. Then, by Theorem 6.4, the map

$$\mathrm{p} \colon \Lambda_f \to S \tag{6.4}$$

is a finite-to-one covering map, and

$$f \colon \Lambda_f \to \Lambda_f \tag{6.5}$$

is a deck transformation interchanging the sheets of this covering. The case of most interest for us will be the case when (6.4) is a diffeomorphism. In that case f has to be the identity mapping, and *all points of* Λ_f *are fixed points of* f. Let us consider the corresponding situation for Λ_{f_1}: Since Λ_f is diffeomorphic to Λ_{f_1}, Theorem 6.4 (applied to Λ_{f_1}) implies that

$$\mathfrak{p}\colon \Lambda_{f_1} \to S \tag{6.6}$$

is a diffeomorphism. On the other hand since Λ_{f_1} and $f_1(\Lambda_{f_1})$ are isodrastic, there exists a function

$$\phi\colon \Lambda_{f_1} \to \mathbb{R} \tag{6.7}$$

(the defining function of $f(\Lambda_{f_1})$ with respect to a tubular neighborhood of Λ_{f_1}) such that if p is a critical point of ϕ, then Λ_{f_1} and $f(\Lambda_{f_1})$ intersect at p. For such a point

$$\mathfrak{p}(p) = \mathfrak{p}(f(p));$$

so the fact that the map (6.6) is a diffeomorphism implies that $p = f(p)$; i.e., every such point is a fixed point of f_1. Thus we have proved:

THEOREM 6.7. *The function*, (6.7), *has the property that each of its critical points is a fixed point of* f_1.

Note, in particular, that since Λ_f and Λ_{f_1} are C_1-close, and *every* point of Λ_f is a fixed point of f, there exists a fixed point of f which is "close" to each of the fixed points of f_1 whose existence is predicted by Theorem 6.7.

6. Let us now suppose that the system (2.1) is integrable. Let $(I_1, ..., I_n, \theta_1, ..., \theta_n)$ be action–angle coordinates on W, let T^n be the n-torus associated with the θ's, and let

$$\mathfrak{p}\colon W \to T^n \tag{6.8}$$

be the mapping which maps the point in W with coordinates (I, θ) onto the point in T^n with coordinates, θ. Let p_0 be a periodic point of the system (2.1) of period N and let $r_1, ..., r_n$ be the winding numbers of the trajectory through this point. Then the torus

$$\Lambda = \{p \in W, I(p) = I(p_0)\} \tag{6.9}$$

consists entirely of periodic points of period N, and for every point on this torus the trajectory through this point also has $r_1, ..., r_n$ as winding numbers. Suppose that f satisfies the non-degeneracy condition (5.10). Then the set (6.9) is an isolated component of the fixed point set of f^N; so if we replace the domain of definition of the I_k's by a small subdomain, we can arrange for Λ to be the set of *all* periodic points of f of period N. Hence, letting $f = f^N$ in Theorem 6.5, we obtain

$$\Lambda = \Lambda_{f^N}. \tag{6.10}$$

Now let $f_1 \colon W \to W$ be another symplectic mapping of W onto itself, and let

$$f_1^k, \qquad k = 0, \pm 1, ..., \tag{6.11}$$

be the dynamical system that it generates. If f and f_1 are isodrastic, then (by Theorem 6.3) so are f^N and f_1^N. Suppose, in addition, that f^N and f_1^N are "C_1-close." Then, as a corollary of Theorem 6.6 we obtain:

THEOREM 6.8 (Birkhoff and Lewis). *There exist at least two periodic trajectories of the system* (6.11) *of period N and with $r_2, ..., r_n$ as winding numbers. Moreover, these trajectories are "close" to periodic trajectories on Λ of the unperturbed system.*

Remark. In the applications which we will discuss below, N tends to infinity. We will therefore have to worry a good deal about whether f^N and f_1^N are sufficiently close for us to be able to use this result.

7. The Birkhoff Canonical Form

Let X be a symplectic manifold, f a symplectic mapping of X into itself, and p a fixed point of f. Suppose that the eigenvalues of df_p are all of modulus one, that is, of the form

$$e^{\pm 2\pi i \omega_k}, \qquad k = 1, ..., n \tag{7.1}$$

and suppose in addition that the numbers,

$$\omega_0, ..., \omega_n, \qquad \omega_0 = 1$$

are independent over the rationals (i.e., if

$$a_0\omega_0 + \cdots + a_n\omega_n = 0,$$

and the a_i's are rational, then they are all equal to zero).

Then one can find a system of Darboux coordinates centered at p such that, in these coordinates, the Taylor series of the mapping f has a very simple form, which we will describe below. For the moment let

$$x_1, ..., x_n, \qquad y_1, ..., y_n$$

be any Darboux coordinate system centered at p, and let r_i and θ_i be the polar coordinates associated with x_i and y_i; i.e.,

$$x_i = r_i \cos \theta_i \qquad \text{and} \qquad y_i = r_i \sin \theta_i.$$

Then

$$\omega = \sum dx_i \wedge dy_i = \sum r_i dr_i \wedge d\theta_i;$$

so if we set $2s_i = r_i^2$, the θ_i's and s_i's are again a Darboux coordinate system.

THEOREM 7.1 (Birkhoff, Lewis, and Sternberg). *There exist a Darboux coordinate system centered at p and a C^∞ function, $F = F(s_1, ..., s_n)$, with $F(0) = dF(0) = 0$, such that in terms of this coordinate system F has the form*

$$\begin{aligned} f^* s_i &= s_i + O(|s|^\infty) \\ f^* \theta_i &= \theta_i + \omega_i + \frac{\partial F}{\partial s_i} + O(|s|^\infty). \end{aligned} \tag{7.2}$$

Proof. See [St].

Another way of stating this result is that the symplectic mapping, f, has infinite order of contact at $s = 0$ with the mapping, f_1 of (s, θ) space onto itself defined by

$$\begin{aligned} f_1^* s_i &= s_i, \\ f_1^* \theta_i &= \theta_i + \omega_i + \frac{\partial F}{\partial s_i}. \end{aligned} \tag{7.3}$$

Note, however, that the dynamical system associated with the mapping (7.3) is completely integrable; the s's and θ's are action and angle coordinates for this system, so the theorem says that if the eigenvalues of df_p have the properties described above, the system (7.2) has infinite order of contact with the *integrable* system (7.3) at p.

The map, f, is said to satisfy the *Birkhoff twist condition* at the point, p, if

$$\det\left(\frac{\partial^2 F}{\partial s_i \partial s_j}(0)\right) \neq 0.$$

If this condition holds we can, without loss of generality assume that

$$\det\left(\frac{\partial^2 F}{\partial s_i \partial s_j}(s)\right) \neq 0$$

at all points of s-space. In other words we can assume that the integrable dynamical system (7.3) is non-degenerate. This implies (see Section 5) that the function, $F(s)$, is a symplectic invariant of this system. Since (7.2) and (7.3) have infinite order of contact at $s=0$, it follows that all terms in the Taylor series expansion of $F(s)$ at $s=0$ are symplectic invariants of the system (7.2). We will henceforth refer to these invariants collectively as the *Birkhoff canonical form of the system* (7.2).

Let W be the space, $\mathbb{R}^n \times T^n$, associated with the coordinates (s, θ). Since we are only interested in the behavior of f near p we can, without loss of generality, make any modification we want to the mapping, f, outside of a small neighborhood of p. In particular we can assume:

A.1. The symplectic mappings, f and f_1, are globally defined on W.

A.2. f and f_1 are equal outside a small neighborhood of the set $s=0$.

A.3. f and f_1 have infinite order of contact along the Lagrangian manifold, $s=0$.

A.4. There exists a Hamiltonian isotopy

$$f_t\colon W \to W, \qquad 0 \leqslant t \leqslant 1 \tag{7.4}$$

mapping the set $s=0$ onto itself, such that $f_t = f$ when $t=0$ and $f_t = f_1$ when $t=1$.

A.5. The Legendre transform

$$\mathscr{L}_F\colon \mathbb{R}_n \to U \tag{7.5}$$

associated with the function, $F(s)$, is a diffeomorphism of $\mathbb{R}^n$ onto U.

We recall from Section 5 that if q is a point in U with coordinates

$$q_i = \omega_i + 2\pi \frac{r_i}{N}, \qquad i = 1, ..., n, \tag{7.6}$$

and s_0 in the point in $\mathbb{R}^n$ which is mapped onto q by $\mathscr{L}_F$, then all points on the fiber above s_0 in W,

$$\Lambda_q = \{(s_0, \theta), \theta \in \mathbb{T}^n\}, \tag{7.7}$$

are periodic points (of period N) of the system (7.3), and, for every point on this set, the trajectory through this point has winding numbers $r_1, ..., r_n$ with respect to $d\theta_1, ..., d\theta_n$. Our objective for the moment is to prove:

THEOREM 7.2. *Let C be a positive constant and k a large positive integer. Then there exist a positive constant C_1 and a positive integer N_o (depending on C and k) such that if $r_1^2 + \cdots + r_n^2 \leqslant C$ and $N > N_0$, there exists a periodic* *point, p, of the system (7.2) which is of period N and has the property that it and all its iterates are a distance less than $C_1 N^{-k}$ from the set (7.7). Moreover, the trajectory through p has the same winding numbers, viz., $r_1, ..., r_n$, as the trajectories on Λ_q of the unperturbed system.*

Proof (in Five Easy Steps)

LEMMA 1. *For every $C>0$ and every positive integer k there exists a constant, C_k, such that if $p=(s, \theta)$, with $|s| \leqslant C/N$, and $(s_r, \theta_r)=f^r(p)$, then for $r \leqslant N$*

$$|s_r - s| \leqslant C_k N^{-k}. \tag{7.8}$$

Proof. Left to the reader. (Also see the Appendix.)

LEMMA 2. *For every constant, $C>0$, and every positive integer, k, there exists a constant, C_k, such that if $p=(s, \theta)$ with $|s| \leqslant C/N$ and $(s_r, \theta_r)=f^r(p)$, then*

$$\left| r\left(\omega_i + \frac{\partial F}{\partial s_i}(s)\right) - (\theta_r - \theta)\right| \leqslant C_k N^{-k}. \tag{7.9}$$

Let Y be the submanifold of $W \times W$ consisting of the points (s, θ, s', θ') with $\theta = \theta'$ mod $2\pi\mathbb{Z}$.

LEMMA 3. *Given $C>0$ there exists an N_0 such that if $N>N_0$ and* graph *f^N intersects Y at $(s, \theta, s_N, \theta_N)$ with $|s| \leqslant C/N$, then* graph f^N *and Y intersect* transversally *at this point.*

LEMMA 4. *Given a constant $C>0$ and a (large) positive integer k, there exists an integer, N_0, and a constant C_k such that if q is the point* (7.6) (with *$\sum r_i^2 < C$ and $N>N_0$) and s_0 is its pre-image, then the set of points, (s, θ), in W satisfying*

$$(f^N)^* \theta_i = \theta_i \quad \text{mof } 2\pi\mathbb{Z}, \quad \textit{and} \quad |s - s_0| \leqslant C_k N^{-k} \tag{7.10}$$

is an n-dimensional submanifold, Λ, of W which is mapped diffeomorphically onto T^n by the map $(s, \theta) \to \theta$.

Proof. We leave the implications, Lemma 1 ⇒ Lemma 2 ⇒ Lemma 3 ⇒ Lemma 4 as exercises. (See also the Appendix.)

Since f can be deformed into f_1 by a Hamiltonian isotopy and f_1 can be deformed into the identity by a Hamiltonian isotopy, the Lagrangian submanifolds, Λ and $f(\Lambda)$, of W are isodrastic in the sense of Section 6.

Moreover, Λ and $f(\Lambda)$ are C^1-close; so, by Theorem 6.8, Λ contains a fixed point of the mapping, f^N (i.e., a periodic point of period N of the system (7.2)). By Lemma 2 this point and its iterates are a distance $< C_k N^{-k}$ from the set (7.7); in fact, there exists a point, p_1, on this set so that it and its iterates (with respect to f_1) are a distance $< C_k N^{-k}$ from p and its iterates with respect to f. However, p_1 is a periodic point of the unperturbed system of period N, and the trajectory through p_1 has winding numbers $r_1, ..., r_n$. Thus, by the estimates, (7.8) and (7.9), the trajectory through p also has $r_1, ..., r_n$ as winding numbers. Q.E.D.

Now let α be the one-form, $\sum s_i d\theta_i$, and let ϕ and ϕ_1 be the functions defined by the identities

$$f^*\alpha - \alpha = d\phi$$

and

$$f_1^*\alpha - \alpha = d\phi_1.$$

If we normalize ϕ and ϕ_1 by requiring that they be zero at $s=0$ then by (7.2),

$$\phi - \phi_1 = O(|s|^\infty). \tag{7.11}$$

Note by the way (see Section 5) that ϕ_1 is a function of s alone (does not depend on θ). Let p be the periodic point of the system (7.2) predicted by Theorem 7.2 and let $\gamma = \gamma_{N,r}$ be the trajectory through p. Let $P(N, r) = P(\gamma, \alpha)$ be the "period" of this trajectory, i.e., the sum

$$\sum \phi(p_i), \qquad p_i \in \gamma.$$

By the estimates in Theorem 7.2 and by (7.11) this sum differs by an error term of order $O(N^{-k})$ from $N\phi_1(s)$, which is the "period," $P_1(N, r)$, of any of the periodic trajectories of the system, (7.3), sitting over s_0. However, we showed in Section 5 that there exists a smooth function, H, on U such that

$$P_1(N, r) = H\left(\frac{r_1}{N}, ..., \frac{r_n}{N}\right);$$

so we have proved

THEOREM 7.3. *The* "periods" *of the periodic trajectories, $\gamma_{N,r}$, satisfy*

$$P(N, r) = H\left(\frac{r_1}{N}, ..., \frac{r_n}{N}\right) + O(N^{-\infty}) \tag{7.12}$$

as $N \to \infty$ (provided that the r's stay bounded).

In particular, let v be the unit vector

$$\frac{1}{|r|}(r_1, ..., r_n)$$

and let $t_N = \pm r/|r|$. The theorem says that the limit, as $N \to \infty$, of the difference,

$$\frac{H(t_N v) - H(0)}{t_N},$$

of the second difference,

$$\frac{H(t_N v) + H(-t_N v) - 2H(0)}{t_N^2},$$

etc., can be read off from the periodic spectrum. Hence for any "rational" point, v, on the unit sphere, the Taylor series of the function

$$t \to H(tv)$$

can be read off from the period spectrum. Since the set of rational points is dense in the unit sphere, it follows that the Taylor series of H itself can be read off from the period spectrum. However, as we pointed out in Section 5 the function H determines the system (7.3) up to symplectomorphism; so, to summarize we have proved:

THEOREM 7.4. *The labelled period spectrum of the system* (7.2) *determines f, up to symplectomorphism, in a formal neighborhood of $s = 0$.*

Let us now turn to the theorem mentioned in the Introduction. Let M be a $(2n+1)$-dimensional manifold; let α be a contact form on M; let $\exp t\Xi$, $-\infty < t < \infty$, be the dynamical system associated with this contact form, and let γ be a periodic trajectory of this system. Suppose γ is elliptic and non-degenerate. (For the precise definitions see the Introduction.) We want to show that it is impossible to deform the contact manifold, M, in a formal neighborhood of γ without deforming the period spectrum or the eigenvalues of the linear Poincaré map. Let (M, α_s), $0 \leqslant s \leqslant 1$, be such a deformation. By the implicit function theorem the dynamical system associated with (M, α_s) has a unique periodic trajectory γ_s, $0 \leqslant s \leqslant \varepsilon$, which depends smoothly on s and is equal to γ when $s = 0$. We want to show that if the dynamical system associated with (M, α_s) has the same "spectral" data as the dynamical system, $\exp t\Xi$, then there exists an isotopy, $h_s: M \to M$, depending smoothly on s, such that h_0 is the identity and $h_s^* \alpha_s$ agrees to infinite order with α along γ. Without loss of generality we can assume that $\gamma_s = \gamma$. Fix a point $p \in \gamma$ and let W be a $2n$-dimensional submanifold of M which intersects γ transversally at p. If we choose W small enough, then there is a well-defined Poincaré map

$$f_s \colon (W, p) \to (W, p)$$

associated with the contact system, (M, α_s), for all $s \in [0, \varepsilon)$. By Proposition 2.1 the period spectrum of this map is independent of s; so, by Theorem 7.4, there exists a symplectic mapping

$$g_s \colon (W, p) \to (W, p) \tag{7.13}$$

such that $f_s \circ g_s$ and $g_s \circ f$ have identical Taylor series expansions at p. We claim that it is possible to make g_s depend smoothly on s. In fact suppose g is a symplectic map of (W, p) into (W, p) which *commutes* with the mapping (7.3). It is easy to see that any such map has to be of the form

$$g^* s_i = s_i$$

(and)

$$g^* \theta_i = \theta_i + \frac{\partial G}{\partial s_i}(s),$$

where $G(s)$ is a smooth function of the s's. Therefore, if we require the g_s in (7.13) to satisfy $g_s^* \theta_i = 0$, $i = 1, ..., n$, on the set $\theta_1 = \cdots = \theta_n = 0$, this makes the solution of the conjugacy problem $f_s \circ g_s = g_s \circ f_s$ *unique* (and it is obvious that this solution depends smoothly on s).

Finally, to conclude the proof we note that, by Theorem 3.7, the mapping g_s extends *canonically* to a mapping

$$h_s \colon (U, \gamma) \to (U_s, \gamma_s)$$

satisfying $h_s^* \alpha_s = \alpha$, where U and U_s are "formal" neighborhoods of γ and γ_s in M.

APPENDIX

0. *Introduction*

We consider $\mathbb{R}^{2m}$ equipped with coordinates $(x_i\, y_i)$, $i = 1, ..., m$, which we will denote by (x, y) for short, and with the symplectic form

$$\Omega = \sum dx_i \wedge dy_i = dx \wedge dy.$$

We introduce the subspaces of codimension two, $A_i = \{(x, y) \in \mathbb{R}^{2m}, x_i = y_i = 0\}$ and the open set $U = \mathbb{R}^{2m}/\bigcup A_i$. On the open set U, we use the "action–angle" coordinates (s_i, θ_i), denoted by (s, θ), for short, defined by

$$s_i = (x_i^2 + y_i^2)/2, \qquad x_i = (2s_i)^{1/2} \cos \theta_i, \qquad y_i = (2s_i)^{1/2} \sin \theta_i.$$

We let $|s| = s_1 + \cdots + s_m$.
Restricted to U, the symplectic form, Ω, becomes

$$\Omega = \sum ds_i \wedge d\theta_i = ds \wedge d\theta.$$

Consider now the symplectic diffeomorphism

$$F_0\colon U \to U, \qquad F_0\colon (s_i, \theta_i) \to (s_i^1 = s_i, \theta_i^1 = \theta_i + \omega_i + Q_i(s)),$$

where $\partial Q_i/\partial s_j$ is a symmetric matrix so that $\partial Q_i/\partial s_j(0)$ is invertible.

DEFINITION 0-1. We will denote by $F_{\omega,Q,k}$ the set of infinitely differentiable symplectic mappings defined on some fixed neighborhood of the origin in $\mathbb{R}^{2m}$ which are tangent to F_0 to order k $(k > 3)$.

1. *Study of the Iterates of an Element F of $F_{\omega,Q,k}$*

LEMMA 1-1. *If s belongs to an open convex cone, P_1, with closure in $\mathbb{R}_+^m$ and $|s| < a$ is sufficiently small, the mapping, $F\colon (s_i, \theta_i) \to (s_i^1, \theta_i^1)$, satisfies the following estimates uniformly*:

(i) $s^1 = s + O(|s|^{(k+1)/2})$,

(ii) $\theta^1 = \theta + \omega + Q(s) + O(|s|^{(k-1)/2})$.

Proof. Part (i) is obvious from the fact that f is tangent to F_0 to order k. We obtain (ii) from $\theta^1 = \operatorname{Arccos}(x^1/(2s^1)^{1/2})$ and the fact that s belongs to a convex cone P_1 of $\mathbb{R}_+^m$ implies that for all $i = 1, ..., m$, $1/s_i$ is of order $O(|s|^{-1})$.

LEMMA 1-2. *If P_0, P_1* (resp. P_1, P_2) *are two open convex cones with closure in $\mathbb{R}_+^m$, so that $\mathbb{R}_+^m \supset P_1 \supset P_0$*(resp. $\mathbb{R}_+^m \supset P_2 \supset P_1$), *there is a constant α_1 (resp. α_2) so that for any u in P_0 (resp. P_1), $|u - v| < \alpha_1 |u|$ (resp. $|u - v| < \alpha_2 |u|$) implies that v belongs to P_1 (resp. P_2).*

LEMMA 1-3. *We obtain for the successive iterates of F, $F^j\colon (s_i, \theta_i) \to (s_i^j, \theta_i^j)$ the estimates $(j = 1, ..., N)$*

$$s^j = s + O(N|s|^{(k+1/2)}).$$

LEMMA 1-4. *Let a be such that $a<1$, $Na^{(k-1)/2}<\alpha_2$. If θ belongs to P_1, then we have*

$$\theta^j=\theta+j\omega+jQ(s)+O(N\,|s|^{(k-1)/2}).$$

Proof. By 1-3 and the conditions on a, if s belongs to P_1 then s^j ($j=1, ..., N$) belongs to P_2. We can thus control the angle coordinates as in 1-1.

LEMMA 1-5. *With the same conditions that we had in* 1-4, *we can write*

$$|\partial\theta^j/\partial\theta-\mathrm{Id}|=O(N\,|s|^{(k-1)/2}).$$

This lemma will be used only in the second part.

LEMMA 1-6. *If s belongs to P_1 and $a<1$, $Na^{(k-1)/2}<\alpha_2$, then we have*

$$|\partial\theta^j/\partial s-jQ'(s)|=O(N\,|s|^{(k-3)/2}).$$

Proof. The derivative relative to s is

$$\partial/\partial s=(x/2s)\,\partial/\partial x+(y/2s)\,\partial/\partial y$$

and so the estimate follows from 1-4.

DEFINITION 1-7. An integer N and a multi-index r are said to be *associated* vis-à-vis the convex cone P_0, and the constants β and γ, if the following conditions are satisfied:

(i) $r-N\omega$ belongs to $Q(P_0)$,

(ii) $|r-N\omega|<\beta N^{1/5}$,

(iii) $N>\gamma$.

THEOREM 1-1. *Suppose $s<a$, s belongs to P_1, $a<1$, $Na^{(k-1)/2}<\alpha_2$, and N and v are associated; then the equation*

$$\theta^N=\theta+r$$

can be solved for s in terms of θ,

$$s=\Psi_{r,N}(\theta),$$

where $\Psi_{r,N}$ is a differentiable function defined on the torus. Furthermore, we have the estimate (with $\bar\omega_{r,N}=Q^{-1}(N^{-1}r-\omega)$)

$$\Psi_{r,N}=\bar\omega_{r,N}+O(\bar\omega_{r,N}^{(k-1)/2}).$$

Proof. We write

$$\theta^N = \theta + N\omega + NQ(s) - G_N(s, \theta)$$

and we consider the mapping

$$f: s \to f(s) = \bar{\omega}_{r,N} + Q^{-1}(N^{-1}G_N(s, \theta)).$$

We claim first that f is a strict contraction on any compact set contained in the domain in which s varies.

$$|f(s) - f(s')| < |s - s'| \sup |\partial/\partial s Q^{-1}(N^{-1}G_N(s, \theta))|$$

and by 1-6

$$|(\partial\theta^N/\partial s - NQ'(s)| = O(N|s|^{(k-3)/2}) = |\partial/\partial s G_N(s, \theta)|$$

$$\partial/\partial s Q^{-1}(N^{-1}G_N(s, \theta)) = (Q^{-1})' (N^{-1}G_N(s, \theta) N^{-1}\partial/\partial s G_N(s, \theta)$$
$$= O(|(s(|^{(k-1)/2}) O(|s|^{(k-3)/2}).$$

If a is small enough, we clearly have

$$|f(s) - f(s')| < \kappa |s - s'| \qquad \text{with} \quad \kappa < 1.$$

Let B be the ball centered at $\bar{\omega}_{r,N}$ of radius $\alpha_1 \bar{\omega}_{r,N}$. We show now that f leaves B invariant.

From condition (ii) in Definition 1-7, we know that

$$|\bar{\omega}_{r,N}| < Q^{-1}(N^{-1}r - \omega)| = O(N^{-4/5}).$$

Now $\frac{4}{5} > 2/(k-1)$ because $k > 3.5$. Since $a = O(N^{-2/(k-1)})$ we can assume that γ is large enough that

$$|\bar{\omega}_{r,N}| < a/2.$$

We can assume also that

$$(1 + \alpha_1) < 2$$

so that if s belongs to B, we obtain

$$|s - \bar{\omega}_{r,N}| < \alpha_1 |\bar{\omega}_{r,N}|$$
$$|s| < (1 + \alpha_1)|\bar{\omega}_{r,N}| < a.$$

Furthermore, by condition (i) of Definition 1-7 and Lemma 1-2, $|s - \bar{\omega}_{r,N}| < \alpha_1 |\bar{\omega}_{r,N}|$ implies that s belongs to P_1. We can therefore apply all the preceding majorations to any element s in B. By 1-4 and $|s| < (1+\alpha_1)|\bar{\omega}_{r,N}|$ we obtain the majoration

$$|f(s) - \bar{\omega}_{r,N}| = O(\bar{\omega}_{r,N}^{(k-1)/2}) < \kappa\, |\bar{\omega}_{r,N}|,$$

which proves that f leaves B invariant. The fixed point theorem gives the existence of $\Psi_{r,N}$, and the estimate above implies

$$|\Psi_{r,N}(\theta) - \bar{\omega}_{r,N}| = O(\bar{\omega}_{r,N}^{(k-1)/2}).$$

2. *Existence of Periodic Points*

We now follow a well-known method to prove the existence of periodic points (cf. [F], [M]). We obtain a more meticulous proof of Theorem 6.8 with the estimates that are required.

Let us denote by U_N the set defined by

$$U_N = \{s < a, p \text{ belongs to } P_1, Na^{(k-1)/2} < \alpha_2\}.$$

We consider the product manifold $(U_N \times \mathbb{T}^m) \times (P_2 \times \mathbb{T}^m)$ equipped with the symplectic form

$$\sigma = ds \wedge d\theta - ds' \wedge d\theta'.$$

Let Δ_N be the graph of the mapping F^N. The fact that F^N is symplectic is equivalent to

$$\sigma \mid \Delta_N = 0.$$

To parametrize the graph Δ_N, we can use either the independent coordinates (s, θ) or the coordinates (s, θ') thanks to the following:

PROPOSITION 2-1. *When s is fixed in U_N, $\theta \to \theta^N$ is a diffeomorphism of the torus.*

Proof. By Lemma 1-5, the mapping $\theta \to \theta^N$ is locally a diffeomorphism. Therefore, since the torus is compact, it is a covering map. Now it is easy to see that it is homotopic to the corresponding mapping for F_0, which is a diffeomorphism, hence it is a diffeomorphism.

Let us consider the one-form $\theta\, ds + \theta'\, ds'$. It is closed on the graph, so there is a function S_N defined on the universal cover of the graph so that $\theta\, ds + \theta'\, ds' = dS_N(s, \theta')$. We introduce the function

$$S'_N(s, \theta') = S_N(s, \theta') - s\theta'.$$

PROPOSITION 2-2. *The function $\theta' \to S'_N(s, \theta')$ is periodic of period 2π and hence it is defined on the torus.*

Proof. Let Γ be a loop of index η obtained by varying θ', with s fixed, in the graph Δ_N. Then

$$\int_\Gamma \delta S'_N = \int_\Gamma (\partial S'/\partial\theta')\,d\theta' = \int_\Gamma (s^N - s)\,d\theta'.$$

The one-form $s^N\,d\theta' - sd\theta$ is analytic at the origin. The loop Γ can be contracted to a point in the domain of analyticity of the form and so the integral $\int_\Gamma s^N\,d\theta' - sd\theta$ is zero.

When the angles variables, θ', vary through $2\pi\eta$ radians, so do the angle variables, θ, because the difference $|\theta^N - \theta - N\omega NQ(s)|$ can be supposed smaller than an integer. Thus, we find

$$\int_\Gamma s^N\,d\theta' = \int_\Gamma sd\theta = s\eta = \int_\Gamma sd\theta' \qquad \text{and so} \qquad \int_\Gamma dS'_N = 0.$$

THEOREM 2-3. *If the multi-index r and the integer N are associated, the symplectic mapping F has periodic points of period N and of index r relative to the axes A_i. The action coordinates of these periodic points belong to U_N and so they satisfy the estimate of Theorem* 1-1.

Proof. We consider the set $\{\theta^N = q + r\}$ in the graph Δ_N. By Theorem 1-1, it can be parametrized by $s = \Psi_{r,N}(\theta)$, and so it is a torus, which we will denote by $\mathbb{T}_{r,N}$. We introduce, on this torus, the function $R_{r,N}$ defined as

$$R_{r,N}(\theta') = S'_N(\Psi_{r,N}(\theta'), \theta').$$

We can check that

$$\partial R_{r,N}(\theta')/\partial\theta' = s^N(s, \theta) - s.$$

The equation for the periodic points

$$s^N(s, \theta) - s = 0, \qquad \theta^N = \theta + r$$

is equivalent to

$$\partial R_{r,N}(\theta')/\partial\theta' = 0,$$

which always has a solution because a differentiable function always has two critical points on a torus.

3. *Control of the Period Spectrum*

Let the function f be defined by the equation

$$F^*(\theta ds) - \theta ds = df.$$

Recall from Section 2 that the "period" $L_{r,N}$ of a periodic orbit of index r and of period N is the sum of the values of f over the points of that orbit. We denote by f_0 and $L^0_{r,N}$ the corresponding quantities for the mapping F_0.

THEOREM 3-1. *There is an estimate*

$$|L_{r,N} - L^0_{r,N}| = O(N^{-4k/5+12/5}).$$

Proof. Let us denote by $(x^1, y^1), ..., (x^N, y^N)$ a periodic orbit for F of the type we considered in Proposition 2-1. Let (x_0^1, y_0^1) be a point whose action coordinates are $\bar{\omega}_{r,N}$ and denote by $(x_0^1, y_0^1), ..., (x_0^N, y_0^N)$ the corresponding periodic orbit for F_0. We have to estimate the difference

$$\left|\sum f(x^j, y^j) - \sum f_0(x_0^j, y_0^j)\right|.$$

We can write

$$\left|\sum f(x^j, y^j) - \sum f_0(x_0^j, y_0^j)\right|$$
$$< \left|\sum f(x^j, y^j) - \sum f_0(x^j, y^j)\right| + \left|\sum f_0(x^j, y^j) - \sum f_0(x_0^j, y_0^j)\right|.$$

The action coordinate of (x^1, y^1) can be estimated by Proposition 2-1 as $s = \bar{\omega}_{r,N} + O(\bar{\omega}_{r,N}^{(k-1)/2})$ and by 1-3, the action coordinates of the points of the orbit are estimated by

$$\begin{aligned} s^j &= s + O(N|s|^{(k+1/2)}) = \bar{\omega}_{r,N} + O(N\bar{\omega}_{r,N}^{(k-1/2)}) \\ &= O(N^{-4/5}) + O(N^{-2k/5+1}). \end{aligned}$$

Consider first the quantity $|\sum f(x^j, y^j) - \sum f_0(x^j, y^j)|$. As F is tangent to F_0 to order k at $s = 0$, we conclude that f is tangent to f_0 to order $k+1$. The difference, $f - f_0$, is analytic in a small neighborhood of $s = 0$. If a is small enough the periodic points are inside that neighborhood, and we obtain the estimate

$$\left|\sum f(x^j, y^j) - \sum f_0(x^j, y^j)\right| = O(N^{-2k/5+3/5}).$$

Consider now the quantity $|\sum f_0(x^j, y^j) - \sum f_0(x_0^j, y_0^j)|$. The nice fact about it is that it depends only on the action coordinates of the periodic points. We see right away that $f_0 = cs + K(s)$, where $K' = Q$, and we obtain the estimate

$$\left|\sum f_0(x^j, y^j) - \sum f_0(x_0^j, y_0^j)\right| = O(N^{-2k/5 + 12/5}).$$

4. *Determination of the Infinite Jet of F at the origin by Its Length Spectrum*

Let F be a symplectic mapping defined on a neighborhood of the origin which has the origin as elliptic fixed point.

If we take for F_0 an integrable mapping (see Section 7) with the same Birkhoff canonical form, then F and F_0 are tangent to infinite order at the origin.

THEOREM 4-1. *The period spectrum of F determines the infinite jet of F at the origin.*

Proof. By 3-1, the asymptotic developments of $L_{r,N}$ and $L_{r,N}^0$ are the same. Now the asymptotic development of $L_{r,N}^0$ determines the infinite jet of F_0 because we can put the action coordinate of the sequences of periodic points in any prescribed cone P_0. Hence the same is true of the infinite jet of F.

REFERENCES

[Arn] V. I. ARNOL'D, Mathematical methods of classical mechanics, Springer-Verlag, New York, 1978.

[C_1] Y. COLIN DE VERDIÈRE, Spectre du laplacien et longueurs des géodesiques periodiques, *Comp. Math.* **27** (1974), 159–184.

[C_2] Y. COLIN DE VERDIÈRE, Sur les longueurs des trajectoires periodiques d'un billiard, *Sém. Sud-Rhodanien Geom.* **111** (1983), 122–140.

[Ch] J. CHAZARAIN, Formule de Poisson pour les variétés riemanniennes, *Invent. Math.* **24** (1974), 65–82.

[D] H. DONNELLY, On the wave equation asymptotics of a compact negatively curved surface, *Invent. Math.* **45** (1978), 115–137.

[DeR] G. DERHAM, "Variétés différentiables," Hermann, Paris, 1955.

[DG] J. J. DUISTERMAAT AND V. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, *Invent. Math.* **29** (1975), 39–79.

[Dui] J. J. DUISTERMAAT, On the existence of global action–angle variables, **33** (1980), 687–706.

[ERT] G. ESKIN, J. RALSTON, AND E. TRUBOWITZ, On isospectral periodic potentials, I, *Comm. Pure Appl. Math.* **37** (1984), 647–676.

[F] J. P. FRANCOISE, Proprêtés de généricité des transformations canoniques, *in* "Lecture Notes in Mathematics, Vol. 1007," pp. 216–259, Springer-Verlag, New York/Berlin, 1984.

[GM] V. Guillemin and R. Melrose, A cohomological invariant of discrete dynamical systems, *in* "E. B. Christoffel, the Influence of His Work on Mathematics and Physical Sciences," Birkhäuser, Boston, 1981.

[GS] V. Guillemin and S. Sternberg, "Geometric Asymptotics," Amer. Math. Soc. Surveys, Vol. 14, Amer. Math. Soc., Providence, RI 1977.

[Hör] L. Hörmander, "The Analysis of Linear Partial Differential Operators III," Springer-Verlag, New York, 1985.

[KB] A. Katok and K. Burns, Manifolds with non-positive curvature, *Ergodic Theory Dynamical Systems* **5** (1985), 307–317.

[L] R. de la Llave, Invariants for smooth conjugacy of hyperbolic systems, *Comm. Math. Phys.* **109**, No. 3 (1987), 369–378.

[LMM] R. de la Llave, J. M. Marco, and R. Moriyon, Canonical perturbation of Anosov systems and regularity results for the Livsic cohomology equation, *Ann. of Math.* **123** (1986), 537–611.

[MM] S. Marvizi and R. Melrose, Spectral invariants of convex planar regions, *J. Differential Geom.* **17** (1982), 475–502.

[M] J. Moser, Proof of a generalized form of a fixed point theorem, *in* "Lecture Notes in Mathematics, Vol. 597," pp. 464–495, Springer-Verlag, New York/Berlin, 1976.

[O] J. P. Otal, Le spectre marqué des longueurs des surfaces à courbure negative, *Ann. of Math.*, to appear.

[OPS] B. Osgood, R. Phillips, and P. Sarnak, Extremals of determinants of Laplacians, *J. Funct. Anal.* **80** (1988), 148–211.

[Sm] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.* **73** (1967), 747–817.

[St] S. Sternberg, Infinite Lie groups and the formal aspects of dynamics, *J. Math. Mech.* **10**, No. 3 (1981), 451–476.

[W] A. Weinstein, Connections of Berry and Hannay types for moving Lagrangian submanifolds, preprint, June 1988.